An Analysis of the Combinatorial Game Princess and the Roses

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INTRODUCTION AND BACKGROUND

What is a combinatorial game?
•2 players play and alternate turns. The game ends when a player has no legal move.
•*Complete information:* there is no chance – both players know all aspects of the game
•Examples: Tic-tac-toe, checkers, chess

Important facts for analyzing combinatorial games
•When analyzing games, the positions where the 2nd player wins are generally looked at. Players are assumed to play optimally when analyzing games.
•Theorem: The 1st player has a winning strategy if he **can** move to a position where the 2nd player has a winning strategy. The 2nd player wins if the 1st player **must** move to a position where the 1st player has a winning strategy.

THE GAME

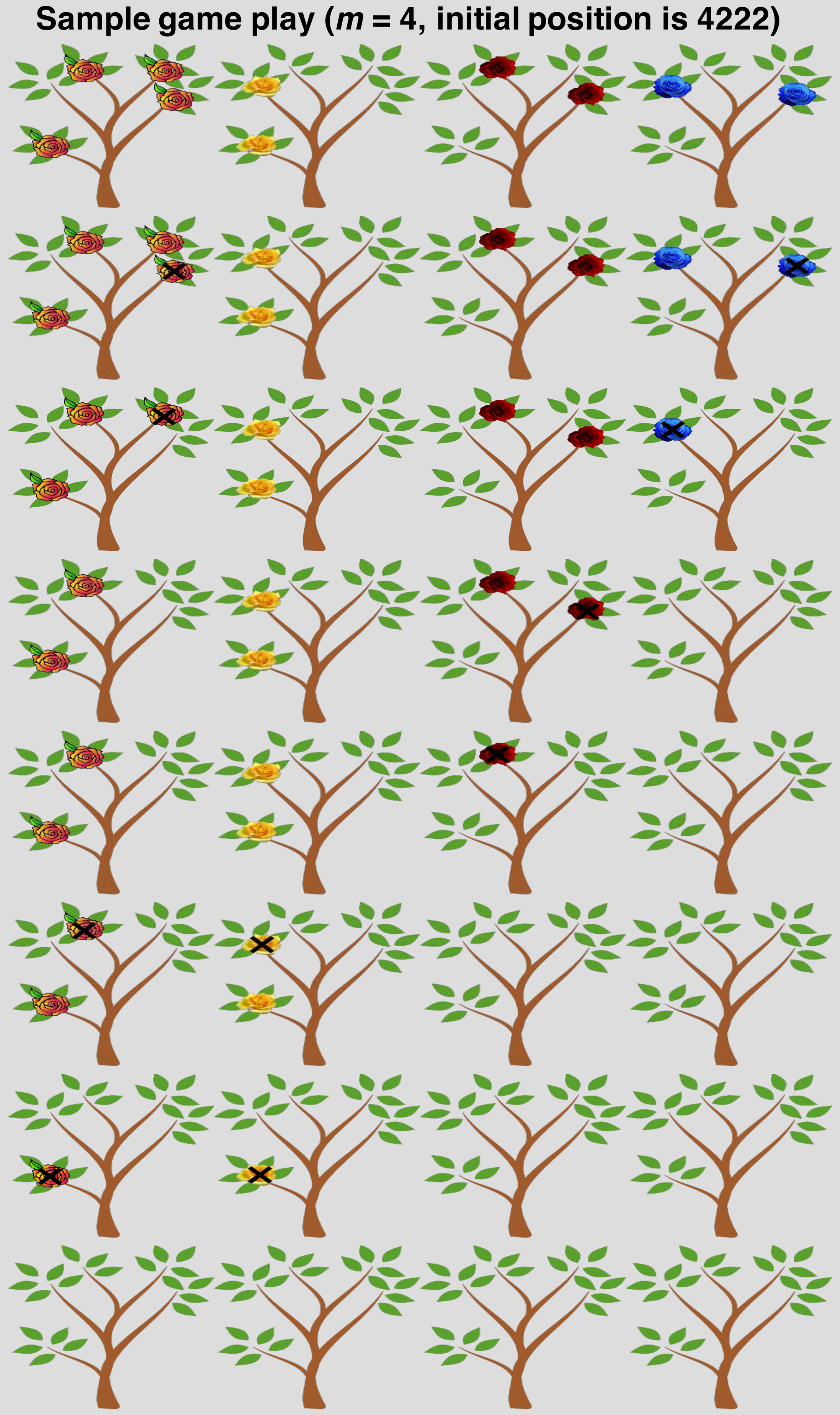
A princess has two suitors. Every day each suitor brings the princess either one rose or two roses, but never two roses from the same bush. The last suitor to bring the princess a flower will win the princess.

More mathematically: There are some number of heaps (m), each containing some number of chips. A turn consists of taking exactly one chip from either one heap or two heaps. The last person to move wins.

For notation, if a game has heaps of size 3, 4, 4, and 2, the heaps are put in decreasing order and the game will be written as 4432. In addition, there are m heaps.

The variant: Instead of taking from up to 2 heaps on a turn, a player may take from up to $m - k$ heaps on his turn where k is a constant.

Sample game play ($m = 4$, initial position is 4222)



Initially, there are 4 bushes with 4, 2, 2, and 2 roses respectively. The position is 4222.

The 1st player moves, taking one rose from the 1st bush and one from the 4th leaving the position 3221.

The 2nd player takes 1 rose from the 1st bush and one from the 4th bush on his move leaving 2220.

The 1st player takes 1 rose from the 3rd bush leaving 2210.

The 2nd player takes 1 rose from the 3rd bush leaving 2200.

The 1st player takes 1 rose from the 1st bush and 1 from the 2nd bush leaving 1100.

The 2nd player takes 1 rose from the 1st bush and 1 from the 2nd bush leaving 0000.

There are no remaining moves for the 1st player, meaning that the 2nd player wins.

RESEARCH OBJECTIVES AND METHODS

Past results by previous researchers:

- Princess and the Roses has been solved for up to 5 heaps ($m = 5$)
- These solutions are solely determined by the ordering of the heaps and the parity of the number of chips in each heap, or whether the number is even or odd.
- However, it has been found that parity does not determine the solution for $m > 5$
- Some progress has been made where $m = 6$ and the last heap has 1 chip.
- The variant seems largely unstudied except for the case where $k = 0$.

Objectives:

- Solve the variant for up to $k = 3$
- Verify past results found for Princess and the Roses
- Determine why parity fails for larger cases and continue the analysis of $m = 6$.

Methods:

- A computer program that found 2nd player win positions was designed.
- Patterns in the data found by the program were used to form conjectures about the general strategy.
- These conjectures were attempted to be proven.

RESULTS

Why parity stops determining the solution:

Consider 2 piles of size 4 in a row: 44. Note that this can only move to 43 and not 34 because the sizes of the heaps must remain in descending order.

In general, 2 even sized heaps (EE) cannot always move to OE because the 2 even heaps could have the same size. Likewise, 2 odd sized heaps (OO) cannot always move to EO because the 2 odd heaps could have the same size.

This one exception is enough to cause the general solution to not rely on parity.

Variant: 2nd player win positions

E represents an even number, O an odd number, and heaps must be listed in descending order by size.

	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
$k = 0$	EE….EE (all heaps even)						
$k = 1$	All positions	EEE…E (all even) or OOO…O (all odd)					
$k = 2$	All positions		EEE, OOE, EOO, OEO	EE…E (all even) and OO…OE (all odd except for the last heap is even)			
$k = 3$	All positions			EEEE, OOEE, OOOO, EOOE, EEOO, OEOE, OEEO, EOEO	EEEEE, OOOOE, EOOOO, OEEOO	EEEEEE, OOOOEE, EEOOOO, OOEEOO	Not determined by parity, unsolved – see section

When $k \le m$, $m - k \le 0$, meaning that there are no moves for the 1st person available, so the 2nd player automatically wins.

When $k = m + 1$, $m - k = 1$, so a player takes exactly 1 chip on each turn. Therefore, the 2nd player wins when the total number of chips is even.

When $k = m + 2$, the game is Princess and the Roses.

Relationship between m and k:

Any parity can be reached from EEE…EE and OO…OEE..E where there are $k - 1$ E's if the requirement that the heaps stay in the correct order is ignored for $m \ge 3k - 2$.

This has the important implication can that the only source of 2nd player win positions besides EE…EE and OO..OEE…E for $m \ge 3k - 2$ is the exception from EE not being able to go to OE when the 2 E's are equal.

RESULTS CONTINUED

Variant $k = 3$, $m = 7$

Not determined by parity – Note that EEEEEEE and OOOOOEE can reach any parity combination.

However, consider 3333221 which cannot move to 3333120 because the ordering is wrong, so it cannot move to EEEEEEE or to OOOOOEE. Problems such as this creates a chain of 2nd player win positions that differ from the expected. Example:

3333221 is a 2nd player win position →	Because it cannot move to 3333120 since the 2's are equal
3333222 is a 1st player win position →	This is OOOOOEE so would normally be a 2nd player win, but it can now move to 3333221.
3333333 is a 2nd player win position	This is OOOOOO which would normally be a 1st player win, but it can no longer move to 3333222 because that is no longer a 2nd player win.

Results found:

All 2nd player win positions are in the form EEEEEEE, OOOOOEE, OOOOEEO, OOOOOOO, or OOOOEOE, but it is not fully determined when each is a 2nd player win position. EEEEEEE is always a 2nd player win position.

CONCLUSION

Main Findings:

•The partial results in the variant for $k = 3$ that show some progress in analyzing games where the solutions are not bound by parity.

•The solutions for smaller k for the variant - this game seems largely unstudied and is a natural extension of Princess and the Roses.

Significance:

•The analysis of the variant can possibly shed light on the normal Princess and the Roses because it shows why and in what ways parity stops determining which positions are 2nd player win positions.

•In addition, the case where $k = 4$ in the variant may be easier to analyze than the case where $m = 6$ in Princess and the Roses, and solutions to other values of m in this variant could be useful in conjecturing patterns in the unsolved version of Princess and the Roses.

•The analysis of Princess and the Roses provides unique insight into impartial games that are difficult to split up into sub-games.

Areas for Further Study:

•Complete the solution for $k = 3$ in the variant

•Analyze $k = 4$ and apply the results to unsolved cases of Princess and the Roses.

•Optimize the computer program used to analyze 2nd player win positions.

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